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J. VAN DE LUNE

A NOTE ON THE ZEROS OF FLETT'S FUNCTION

kruislaan 413 1098 SJ amsterdam

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A note on the zeros of Flett's function

by

J. van de Lune

ABSTRACT

This note contains the description of a method for the numerical computation of the real zeros of functions such as $F(z) = \Sigma_{n=1}^{\infty} \frac{1}{n} \sin \frac{z}{n}$. Some real and complex zeros of this particular function are presented.

KEY WORDS & PHRASES: *Special functions*

0. INTRODUCTION

This note is an extended version of a lecture held at the conference: "Numbertheory and Computers", Mathematical Centre, Amsterdam, September, 1980.

We consider (what we call) Flett's function

$$F(z) := \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{z}{n}, \qquad z \in \mathbb{C}.$$

It was observed by HARDY and LITTLEWOOD [2] and FLETT [1] that the O-problem for the restriction of F to the positive real axis is much like the corresponding problem for $\zeta(1+it)$ as $t \to \infty$, where ζ denotes Riemann's zeta-function. This observation, together with the well-known fact that $\zeta(1+it)$ does not vanish for $t \in \mathbb{R}^+$, led us (and others) to the question whether F has any positive zeros and if so, how to locate them.
In Section 1 we show that F has positive zeros indeed and we compute all of them in the interval $(0,2000]$.

From the power series representation

$$F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n+2)}{(2n+1)!} z^{2n+1}, \qquad z \in \mathbb{C}.$$

it is easily seen that the entire function $F(z)/z$ is an even function of order 1, from which it follows (by the general theory of entire functions of finite order (cf. SANSONE and GERRETSEN [5], pp. 323-324)) that F has infinitely many (complex) zeros.
In Section 2 we compute some of the non-real zeros of F.

It may be noted that all zeros found so far lie either on or rather close to the real axis. Since it seems likely that F has infinitely many non-real zeros, this leads us to the (open) question whether the imaginary parts of the zeros of F form a bounded set or not.

1. COMPUTATION OF SOME REAL ZEROS OF F

1.1. The smallest positive zero of F

Since F is an odd function, we may restrict ourselves to the positive real axis.
It is clear that

$$F(t) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{t}{n} > 0 \qquad \text{for } 0 < t \leq \pi.$$

Since

$$F'(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{t}{n} \qquad \text{for all } t > 0,$$

we have

$$\sup_{t>0} |F'(t)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6,$$

so that $|F'(t)|$ is bounded by $M := \pi^2/6$ (< 1.645).
By the mean value theorem for differentiable functions we have for $t \geq t_0 := \pi$ and some $\theta \in (0,1)$

$$F(t) = F(t_0) + (t-t_0)F'(t_0+\theta(t-t_0)) \geq F(t_0) - (t-t_0)M$$

so that $F(t) > 0$ for $t_0 \leq t < t_0 + F(t_0)/M$. Since F is not linear, we may even conclude that

$$(*) \qquad F(t) > 0 \qquad \text{for } t_0 \leq t \leq t_0 + F(t_0)/M.$$

We call this well-known method for extending a given zero-free region the

MAXIMAL SLOPE PRINCIPLE. If in (*) $F(t_0)$ is replaced by a smaller positive number, F^*, say, and M is replaced by a larger number, M^*, say, then the conclusion

$$F(t) > 0 \qquad \text{for } t_0 \le t \le t_0 + F^*/M^*$$

is still true. This observation will be useful later on. In view of the actual implementation of this procedure we write

$$F(t) = S_N(t) + R_N(t)$$

where

$$S_N(t) = \sum_{n=1}^{N} \frac{1}{n} \sin \frac{t}{n} \qquad \text{and} \qquad R_N(t) = \sum_{n=N+1}^{\infty} \frac{1}{n} \sin \frac{t}{n} .$$

From now on we assume that $0 < t/N < \pi$ so that always $R_N(t) > 0$ and hence $F(t) > S_N(t)$. A crude estimate of $R_N(t)$ is given by

$$(0<)R_N(t) < \sum_{n=N+1}^{\infty} \frac{1}{n}\frac{t}{n} = t \sum_{n=N+1}^{\infty} \frac{1}{n^2} < t/N$$

so that

$$(S_N(t)<)F(t) < S_N(t) + t/N.$$

Suppose that $S_N(t_0) > 0$ for some $t_0 > 0$. Then, by the maximal slope principle, we have

$$S_N(t) > 0 \qquad \text{for } t_0 \le t \le t_0 + S_N(t_0)/M$$

since

$$\sup_{t>0} |S'_N(t)| = \sup_{t>0} \left| \sum_{n=1}^{N} \frac{1}{n^2} \cos \frac{t}{n} \right| \le \sum_{n=1}^{N} \frac{1}{n^2} < \pi^2/6 = M.$$

Defining $t_1 := t_0 + S_N(t_0)/M$, we have $S_N(t_1) > 0$, and, applying the maximal slope principle once more it follows that

$$S_N(t) > 0 \qquad \text{for } t_1 \le t \le t_1 + S_N(t_1)/M.$$

Clearly $S_N(t) > 0$ for $0 < t \le \pi$ and $N \ge 2$ so that we may start the above procedure with $t_0 = \pi$ and $N = 10$, say. On a programmable pocket calculator (an HP 41C in our case) we ran the following program

```
LBL FLETT        RCL 00
10 STO 00        ÷
0 STO 03         STO + 03
RCL 01           XEQ DSE 00
R/S              GTO SUM
LBL SUM          RCL 03
RCL 01           RCL 02
RCL 00           ÷
÷                STO + 01
XEQ SIN          GTO FLETT
```

Some comments on the program: t is stored in memory 01, the sum $S_N(t)$ is built up in memory 03 and the loop length N = 10 is controlled by DSE in memory 00. M is stored in memory 02. Before running the program, load: (π>) 3.14 STO 01 and (M<) 1.645 STO 02. Set the calculator in the RADIAN mode and the program may be executed by pressing XEQ(alpha)FLETT(alpha). The calculator successively stopped at the following values t_n of t (so that $S_{10}(t) > 0$, and hence $F(t) > 0$, for $0 < t < t_n$)

n	t_n	n	t_n
0	3.1400		
1	3.9584	11	9.4682
2	4.4009	12	9.7210
3	4.7149	13	9.8030
4	4.9926	14	9.8324
5	5.2802	15	9.8433
6	5.6180	16	9.8475
7	6.0600	17	9.8491
8	6.6883	18	9.8497
9	7.5981	19	9.8499
10	8.7093	20	9.8500

Since, as we see from the table, not much progress is made any more with $N = 10$ we replaced the first line after LBL FLETT by 20 STO 00, i.e. we set $N = 20$. The calculator now stopped at the following values of t_n of t (after resetting N we start counting from $n = 0$ on):

n	t_n	n	t_n
0	9.8499		
1	10.1045	6	10.3462
2	10.2190	7	10.3538
3	10.2792	8	10.3585
4	10.3135	9	10.3615
5	10.3339	10	10.3634

After this run we removed R/S from the program and ran the program without any interruption for about 15 minutes with $N = 50$. We found that $S_{50}(t) > 0$ for $0 < t \leq 18.3941$. Then we set N equal to 100 and found after quite a while that $S_{100}(t) > 0$ for $0 < t \leq 35.20$. Similarly we found that $S_{200}(t) > 0$ for $0 < t \leq 35.58$. Continuing with $N = 300$ it turned out that $S_{300}(t) > 0$ for $0 < t \leq 48$. This last run took about one day (on our HP 41C). From here on we used a faster computer (in our case a CDC-CYBER-6600) and found (in a few seconds)

$N = 1000$	$S_N(t) > 0$	for $0 < t \leq 48.2478$
$N = 2000$	$S_N(t) > 0$	for $0 < t \leq 48.3197$
$N = 3000$	$S_N(t) > 0$	for $0 < t \leq 48.3482$
$N = 4000$	$S_N(t) > 0$	for $0 < t \leq 48.3637$
$N = 5000$	$S_N(t) > 0$	for $0 < t \leq 48.3736$
$N = 10000$	$S_N(t) > 0$	for $0 < t \leq 48.3946$
$N = 20000$	$S_N(t) > 0$	for $0 < t \leq 48.4057$
$N = 50000$	$S_N(t) > 0$	for $0 < t \leq 48.4131$

In order to show that this time we are closing in on a zero of F we computed

$$U_N(t) := S_N(t) + t/N \quad (> F(t))$$

for $t = 48.0(.1)49.0$ and $N = 1000$, yielding

t	$U_{1000}(t)$
48.0	.2365
48.1	.1709
48.2	.1143
48.3	.0673
48.4	.0304
48.5	.0039
48.6	-.0119
48.7	-.0166
48.8	-.0103
48.9	.0072
48.0	.0357

It follows that $F(48.6) < 0$ so that the smallest positive zero of F lies in the interval $(48.4, 48.6)$. More accurate calculations (see the next section) show that this zero is approximately $t = 48.4184536114$.

REMARK. In order to see that F assumes negative values in the interval $(48.0, 49.0)$ it suffices to compute $U_N(t)$ with $N = 79$. Indeed, $U_{79}(48.6) < -.0003$.

1.2. A much faster approach

Extending the numerical computations initiated above one will experience that $F(t)$ is preponderantly positive. This observation may be explained as follows. It is easily seen that $F(t) = O(\log t)$ for $t \to \infty$ (one may also use the O-estimates from [1] or [2]) so that we may consider the Laplace-transform $\phi(s)$ of $F(t)$:

$$\phi(s) := \int_0^\infty e^{-st}F(t)\,dt, \qquad s > 0.$$

Observing that

$$\phi(s) = \int_0^\infty e^{-st} \sum_{n=1}^\infty \frac{1}{n} \sin\frac{t}{n}\,dt = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty e^{-st} \frac{e^{\frac{it}{n}} - e^{-\frac{it}{n}}}{2i}\,dt =$$

$$= \sum_{n=1}^\infty \frac{1}{2ni} \left(\frac{1}{s-\frac{i}{n}} - \frac{1}{s+\frac{i}{n}}\right) = \sum_{n=1}^\infty \frac{1}{1+n^2s^2},$$

and considering

$$\sum_{n=1}^{\infty} \frac{s}{1+(ns)^2}$$

as a lower Riemann approximation of the integral

$$\int_0^{\infty} \frac{dx}{1+x^2}$$

it follows that

$$\lim_{s \downarrow 0} \int_0^{\infty} e^{-st} F(t)\,dt \Big/ \int_0^{\infty} e^{-st}\,dt = \lim_{s \downarrow 0} s\phi(s) = \pi/2,$$

so that $F(t)$ has the positive Laplace-Abel limit $\pi/2$ as $t \to \infty$. In combination with the fact that $F(t)$ is a rather small function on $\mathbb{R}^+$ we have a clear indication that $F(t)$ is preponderantly positive.

So far, our systematic search for real zeros of F consisted mainly of the determination of intervals on which S_N is positive, followed by a check whether in the remaining intervals $U_N(t) := S_N(t) + t/N$ is ever negative. In order to make this procedure as profitable as possible we need

(i) a fast procedure to determine whether $S_N(t)$ is positive on a given interval, and

(ii) a good approximation of $R_N(t)$.

The importance of (ii) is clear and in order to understand the importance of (i) we challenge the reader to decide whether $F(t)$ is ever negative in the interval (760,810). One will experience that for large N it is quite time consuming to evaluate $S_N(t)$ and, in a region where $S_N(t)$ is small, the maximal slope principle works very slow.

We now describe a different procedure which runs considerably faster than the procedure described above[1]. For the sake of brevity we write $f(t)$ instead of $S_N(t)$ and we observe that in the process of building up the sums

$$f(t_0) = \sum_{n=1}^{N} \frac{1}{n} \sin \frac{t_0}{n} \qquad \text{and} \qquad f'(t_0) = \sum_{n=1}^{N} \frac{1}{n^2} \cos \frac{t_0}{n}$$

we may in a relatively cheap way also build up the higher derivatives of $f(t)$. Since

$$|f^{k+1}(t)| = \Big| \sum_{n=1}^{N} \frac{1}{n^{k+2}} \begin{matrix}\sin\\ \cos\end{matrix} \frac{t}{n} \Big| \le \sum_{n=1}^{N} \frac{1}{n^{k+2}} < \zeta(k+2) =: M_k$$

---

[1] For another application see [4].

we have for all $t \geq t_0$ (from the Taylor expansion of $f(t)$)

$$P(k,t_0,t) := \sum_{r=0}^{k} \frac{f^{(r)}(t_0)}{r!}(t-t_0)^r - \frac{M_k}{(k+1)!}(t-t_0)^{k+1} \leq f(t).$$

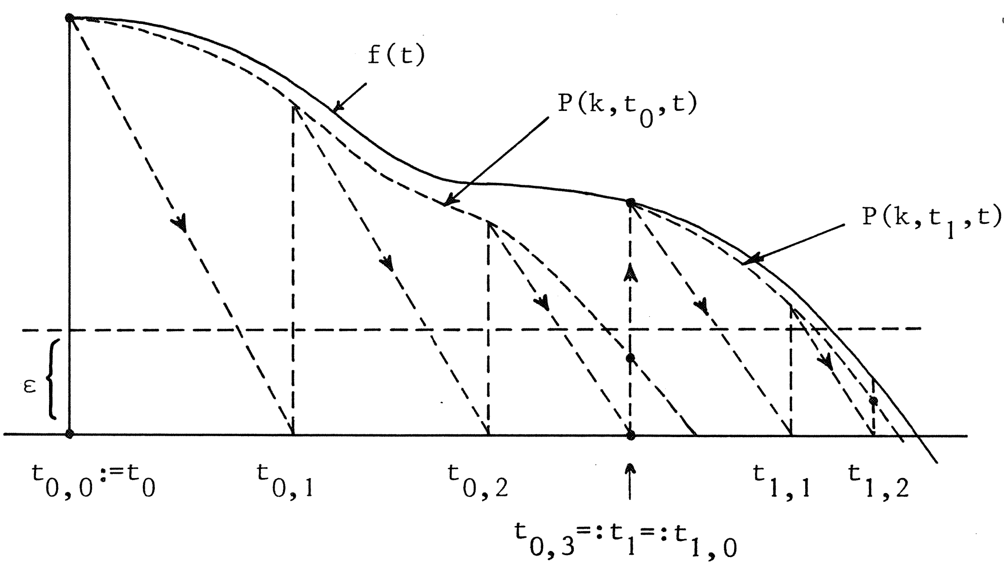
Clearly $f(t) > 0$ (and hence $F(t) > 0$) as long as $P(k,t_0,t) > 0$. Applying the maximal slope principle to the polynomial approximation $P(k,t_0,t)$ of $f(t)$ we avoid a lot of SIN evaluations and thus getting a much faster procedure than the direct application of the maximal slope principle to $f(t)$. In our actual computations we took $k = 10$. Applying the maximal slope principle to $P(k,t_0,t)$ we obtain a sequence $\{t_{0,n}\}_{n=0}^{\infty}$ where

$$t_{0,0} := t_0 \quad \text{and} \quad t_{0,n+1} = t_{0,n} + P(k,t_0,t_{0,n})/M_k \qquad \text{for } n \geq 0.$$

We interrupt the procedure at $t = t_{0,n}$ if $P(k,t_0,t_{0,n}) < \varepsilon(= 10^{-4}$, say). It is easily seen that at such a value of t we still have $f(t_{0,n}) > 0$ so that we may construct a new polynomial

$$P(k,t_1,t) := \sum_{r=0}^{k} \frac{f^{(r)}(t_1)}{r!}(t-t_1)^r - \frac{M_k}{(k+1)!}(t-t_1)^{k+1},$$

where $t_1 := t_{0,n}$, and apply the maximal slope principle to this polynomial as long as $P(k,t_1,t) \geq \varepsilon$, etc. If for some $t_m (= t_{m-1,n})$ we find $P(k,t_m,t_m) < \varepsilon$ then $f(t_m) < \varepsilon$, since $f(t_m) = P(k,t_m,t_m)$. The main features of our procedure may be graphically depicted as follows:



It hardly needs any comment that our procedure may easily be adapted in order to work from the right to the left starting with a t_0 such that $F(t_0) < 0$.

In order to obtain a good approximation of $R_N(t)$ we may use the Euler-MacLaurin summation formula. Writing $M = N + 1$ we have

$$R_N(t) = \sum_{n=N+1}^{\infty} \frac{1}{n} \sin\frac{t}{n} = \sum_{n=M}^{\infty} \frac{1}{n} \sin\frac{t}{n} = \int_{M-0}^{\infty} \frac{1}{x} \sin\frac{t}{x}\, d[x] =$$

$$= \int_{M}^{\infty} \frac{1}{x} \sin\frac{t}{x}\, dx - \int_{M-0}^{\infty} \frac{1}{x} \sin\frac{t}{x}\, d(x-[x]-\tfrac{1}{2}) =$$

$$= \int_{0}^{1/M} \frac{\sin tu}{u}\, du - \int_{M-0}^{\infty} \frac{1}{x} \sin\frac{t}{x}\, d\psi_1(x)$$

where $\psi_1(x) := x - [x] - \frac{1}{2}$.

The first integral can conveniently be written as

$$\int_{0}^{1/M} \frac{\sin tu}{u}\, du = \int_{0}^{1/M} \frac{1}{u} \left(tu - \frac{(tu)^3}{3!} + \frac{(tu)^5}{5!} - + \ldots\right) du =$$

$$= \frac{t}{M} - \frac{1}{3.3!} \left(\frac{t}{M}\right)^3 + \frac{1}{5.5!} \left(\frac{t}{M}\right)^5 - + \ldots$$

so that, taking k terms of this alternating series and choosing $M \geq t/3$ we introduce an absolute error of at most

$$\frac{3^{2k+1}}{(2k+1)(2k+1)!}$$

which, for $k = 13$, does not exceed $2.6*10^{-17}$. The second integral, after repeated integration by parts, is equal to

$$- \int_{M-0}^{\infty} \frac{1}{x} \sin\frac{t}{x}\, d\psi_1(x) = \frac{1}{2M} \sin\frac{t}{M} + \frac{1}{12M^2} \left(\sin\frac{t}{M} + \frac{t}{M} \cos\frac{t}{M}\right) +$$

$$- \frac{1}{720M^4} \left\{\left(6 - \frac{9t^2}{M^2}\right) \sin\frac{t}{M} + \left(\frac{18t}{M} - \frac{t^3}{M^3}\right) \cos\frac{t}{M}\right\} +$$

$$+ \frac{1}{30240M^6} \left\{\left(120 - \frac{600t^2}{M^2} + \frac{25t^4}{M^4}\right) \sin\frac{t}{M} + \left(\frac{600t}{M} - \frac{200t^3}{M^3} + \frac{t^5}{M^5}\right) \cos\frac{t}{M}\right\} +$$

$$+ \int_{M}^{\infty} \psi_6(x) \left\{\left(-\frac{720}{x^7} + \frac{5400t^2}{x^9} - \frac{450t^4}{x^{11}} + \frac{t^6}{x^{13}}\right) \sin\frac{t}{x} + \left(-\frac{4320t}{x^8} + \frac{2400t^3}{x^{10}} - \frac{36t^5}{x^{12}}\right) \cos\frac{t}{x}\right\} dx,$$

where $\psi_6(x)$ is the sixth Bernoulli function defined by

$$\psi_6(x) := \frac{x^6}{720} - \frac{x^5}{240} + \frac{x^4}{288} - \frac{x^2}{1440} + \frac{1}{30240}, \qquad 0 \le x < 1,$$

and

$$\psi_6(x+1) = \psi_6(x), \qquad \text{for all } x \in \mathbb{R}.$$

In order to estimate the last integral

$$I_M(t) := \int_M^\infty \psi_6(x)\{(\ldots)\sin\frac{t}{x} + (\ldots)\cos\frac{t}{x}\}dx$$

we observe that (compare KNOPP [3], pp. 550-552)

$$\sup_{x\in\mathbb{R}} |\psi_6(x)| = \psi_6(0) = \frac{1}{30240},$$

so that

$$|I_M(t)| \le \frac{1}{30240}\{(\frac{720}{6M^6} + \frac{5400t^2}{8M^8} + \frac{450t^4}{10M^{10}} + \frac{t^6}{12M^{12}}) + $$
$$+ (\frac{4320t}{7M^7} + \frac{2400t^3}{9M^9} + \frac{36t^5}{11M^{11}})\}$$

which, for $M \ge \max\{300, t/3\}$ does not exceed $9*10^{-16}$. Since we intend to work in single precision it follows that we need not worry about any serious errors when taking $M \ge \max\{300, t/3\}$ and setting

$$F(t) \simeq \sum_{n=1}^{N} \frac{1}{n}\sin\frac{t}{n} + \sum_{k=0}^{12} (-1)^k \frac{1}{(2k+1)(2k+1)!}(\frac{t}{M})^{2k+1} +$$
$$+ \frac{1}{2M}\sin\frac{t}{M} + \frac{1}{12M^2}(\sin\frac{t}{M} + \frac{t}{M}\cos\frac{t}{M}) +$$
$$- \frac{1}{720M^4}\{(6 - \frac{9t^2}{M^2})\sin\frac{t}{M} + (\frac{18t}{M} - \frac{t^3}{M^3})\cos\frac{t}{M}\} +$$
$$+ \frac{1}{30240M^6}\{(120 - \frac{600t^2}{M^2} + \frac{25t^4}{M^4})\sin\frac{t}{M} + (\frac{600t}{M} - \frac{200t^3}{M^3} + \frac{t^5}{M^5})\cos\frac{t}{M}\}.$$

For F'(t) we derived a similar formula and then applied Newton's method to the intervals containing at least one zero of F. Our numerical results are summarized in the following

TABLE I

All zeros of F in the interval (0,2000]

48.418454	1090.843666
48.766656	1092.143117
123.688980	1123.064213
124.187053	1123.803276
148.138791	1128.108916
149.638442	1129.884077
298.929341	1178.909440
300.455973	1180.265509
336.659318	1278.945576
338.318085	1280.788172
374.317214	1304.951756
375.828297	1305.309292
425.153416	1349.540661
426.108720	1349.660942
487.774018	1354.468455
488.859895	1355.956888
525.223463	1392.884130
526.547965	1393.316410
600.420544	1405.316380
602.070589	1406.094374
651.133817	1430.306042
652.389083	1431.172051
676.015892	1500.068779
677.418529	1500.697985
746.043879	1505.371045
746.724669	1506.596605
751.544138	1556.025180
752.446891	1557.176601
827.225050	1580.995254
827.771829	1581.890319
865.067648	1655.928062
865.623792	1657.737978
877.289535	1693.924478
878.544023	1695.323827
902.137324	1781.920425
903.557222	1783.632227
940.010051	1807.188877
941.335920	1808.066060
952.883980	1844.694996
953.830042	1846.243499
1028.536226	1882.244449
1029.034260	1883.736391
1052.787397	1995.657279
1054.498124	1996.970216

Although we cannot prove that F has infinitely many real zeros, we conjecture that this is actually the case. This is supported by the following heuristical arguments: Let J(N) denote the least common multiple of the first N positive integers. Then

$$F(2\pi J(N)) = \sum_{n=N+1}^{\infty} \frac{1}{n} \sin \frac{2\pi J(N)}{n}$$

which may be expected to be rather small because of the erratic oscillatory behaviour of those terms in the series for F which do not vanish in advance. Moreover,

$$F'(2\pi J(N)) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2\pi J(N)}{n}$$

which may be expected to be about as large as possible, i.e. $\pi^2/6$. Finally, we have

$$F''(2\pi J(n)) = - \sum_{n=N+1}^{\infty} \frac{1}{n^3} \sin \frac{2\pi J(N)}{n}$$

which may be expected to be small again so that F'(t) varies slowly for $t \simeq 2\pi J(N)$. Putting things together we come to the heuristical conclusion that for large N we will probably have a real zero of F in the vicinity of $t = 2\pi J(N)$. Numerical experiments confirm this heuristic prediction as is shown in Table II. The reader will have noticed from Table I that the positive zeros of F come in neighbouring pairs, a phenomenon which persists for the large zeros in Table II (see last column).

TABLE II

N	2πJ(N)	Does F have a (real) zero close to 2πJ(N)?		
1	6.28	NO		
2	12.56	NO		
3	37.69	NO		
4	75.38	NO		
5	376.99	YES at	375.828297	(and 374.317214)
7	2638.93	YES at	2637.376424	(and 2636.565990)
8	5277.87	YES at	5276.243212	(and 5275.545186)
9	15833.62	YES at	15832.481407	(and 15830.638741)
11	174169.82	YES at	174168.677986	(and 174166.977585)
13	2264208.66	YES at	2264207.529272	(and 2264205.609370)
16	4528417.32	YES at	4528416.235171	(and 4528414.175770)
17	76983094.35	YES at	76983093.340652	(and 76983091.078645)

We note that quite often also the integral multiples of the zeros in the second column of this table are fairly good first approximations of further zeros.

We conclude this section with the OPEN QUESTION: Is F bounded below on $\mathbb{R}^+$? We conjecture that it is not.

2.1. Computation of some complex zeros of F

In order to compute a complex zero of F we need a good approximation of F(z) for complex z. The exact representation of F(t) for $t \in R^+$, derived in Section 1, also holds true when the real variable t is replaced by the complex variable z. However, the estimation of the error terms is slightly different.

Approximating the integral

$$\int_0^{1/M} \frac{\sin zu}{u}\, du$$

by

$$\sum_{n=0}^{k} (-1)^n \frac{1}{(2n+1)(2n+1)!} \left(\frac{z}{M}\right)^{2n+1}$$

we introduce an absolute error of at most

$$\sum_{n=k+1}^{\infty} \frac{u^{2n+1}}{(2n+1)(2n+1)!}, \qquad \text{where } u = \frac{|z|}{M} .$$

If, for example we take $M \geq |z|$, this error does not exceed

$$\sum_{n=k+1}^{\infty} \frac{1}{(2n+1)(2n+1)!} < \frac{1}{(2k+3)} \left\{\frac{1}{(2k+3)!} + \frac{1}{(2k+5)!} + \ldots\right\} <$$

$$< \frac{1}{(2k+3)(2k+3)!} \left\{1 + \frac{1}{(2k+4)(2k+5)} + \frac{1}{(2k+5)(2k+6)} + \ldots\right\} =$$

$$= \frac{1 + \frac{1}{2k+4}}{(2k+3)(2k+3)!} < 1.75*10^{-16} \qquad \text{for } k \geq 7.$$

The absolute error due to the deletion of the integral

$$I_M(z) := \int_M^\infty \psi_6(x)\{(\ldots)\sin\frac{z}{x} + (\ldots)\cos\frac{z}{x}\}dx$$

may be estimated by

$$\int_M^\infty \frac{\psi_6(0)}{x^7}\{(720+5400\ |\frac{z}{x}|^2 + 450\ |\frac{z}{x}|^4 + |\frac{z}{x}|^6)\,|\sin\frac{z}{x}| +$$

$$+\ (4320\ |\frac{z}{x}| + 2400\ |\frac{z}{x}|^3 + 36\ |\frac{z}{x}|^5)\,|\cos\frac{z}{x}|\}dx$$

which, again taking $M \geq |z|$, does not exceed

$$\frac{M^{-6}}{6*30240}\{6571\ \frac{e-e^{-1}}{2} + 6756\ \frac{e+e^{-1}}{2}\} < .11*M^{-6}.$$

Hence, taking $M \geq \max\{200,|z|\}$, say, the error will be $< 1.72*10^{-15}$.

Since $F(\bar{z}) = \overline{F(z)}$ and $F(-z) = F(z)$ for all $z \in \mathbb{C}$ and $F(it) \neq 0$ for $t > 0$ we may restrict ourselves to the first quadrant of $\mathbb{C}$. Some (if not all) of the non-real zeros $x+yi$ of F with $x > 0$ and $y > 0$ and $|z| < 501$ are listed in the following

TABLE III

Some zeros $(x,y) := x+yi$ of F

(4.696508, 1.342085)	(155.600477, 1.660510)
(10.923358, 0.774040)	(161.760719, 0.831023)
(17.406469, 1.185511)	(168.210688, 1.120806)
(23.391994, 0.816097)	(174.190019, 0.679102)
(30.056243, 1.240244)	(180.855204, 1.136402)
(35.909567, 0.474922)	(186.677102, 0.398478)
(42.581851, 1.460279)	(193.349796, 1.559899)
(55.040678, 1.291546)	(199.407108, 0.632108)
(61.143027, 1.003766)	(205.884011, 1.283003)
(67.801791, 1.046924)	(211.958643, 0.688488)
(73.537717, 0.046393)	(218.572177, 0.773271)
(80.255168, 1.625391)	(224.277775, 0.426493)
(86.371744, 0.251730)	(231.034973, 1.802581)
(92.753874, 0.993494)	(237.221002, 0.680168)
(98.753072, 0.980764)	(243.614607, 0.749058)
(105.473227, 1.389159)	(249.524593, 0.479778)
(111.353753, 0.272503)	(256.219230, 1.305712)
(117.979606, 1.300330)	(262.118391, 0.710000)
(130.423387, 1.454854)	(268.784096, 1.492749)
(136.584906, 1.119612)	(274.781696, 0.249728)
(143.258458, 0.814422)	(281.234677, 1.353252)

TABLE III (cont'd)

(287.363868, 1.002897)	(400.329985, 0.681712)
(294.043621, 0.793073)	(407.002021, 1.394821)
(306.398514, 1.627776)	(412.917061, 0.936434)
(312.531550, 0.857736)	(419.601418, 1.572956)
(318.976731, 1.300064)	(432.054937, 1.015207)
(325.015798, 0.973798)	(438.106188, 0.621272)
(331.721188, 1.080143)	(444.752434, 0.951640)
(344.115440, 1.316057)	(450.502331, 0.577563)
(350.125201, 0.699738)	(457.233420, 1.779870)
(356.642781, 1.578031)	(463.386105, 0.747243)
(362.803862, 1.125769)	(469.781844, 1.033373)
(369.452936, 0.727577)	(475.761264, 0.810406)
(381.811216, 1.649432)	(482.477017, 1.212181)
(387.979176, 0.581187)	(494.950352, 1.278982)
(394.395834, 0.886020)	(500.919632, 0.313653)

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